§ 2.5 Discrete valuation rings.

discrete valuation on a field :

$$v : K \longrightarrow \mathbb{Z} \cup \{\infty\}.$$

- $v(ab) = v(a) + v(b)$
- $v(a+b) \geq \min(v(a), v(b))$
- $v(a) = \infty \Leftrightarrow a = 0.$

$$\mathcal{O}_K := \{ r \in K \mid v(r) \geq 0 \} \quad (\underline{\operatorname{im} v \neq \{0, \infty\}})$$

$\hookrightarrow$ DVR

$$\mathfrak{m}_K := \{ r \in K \mid v(r) > 0 \}$$

Example: $\mathbb{Z}_p$, $k[[t]]$, ...

Lem: R = domain. TFAE

1) R = DVR
2) $\exists$ irr. element $t \in R$ s.t. $\forall z \in R$ $\quad z \overset{!}{=} ut^n$ for some $u \in R^\times$, $n \in \mathbb{N}$.
3) R = noeth + local + principal max. ideal.

n → order of z

t → uniformizing parameter

$$R = \{ z \in K \mid \operatorname{ord}(z) \geq 0 \}$$
$$\mathfrak{m} = \{ z \in K \mid \operatorname{ord}(z) \geq 1 \}$$

§ 2.6 Forms.

R = domain, $F \in R[X_1, \cdots, X_{n+1}]$ is a form of deg d

$F_* := F(X_1, \cdots, X_n, 1) \in R[X_1, \cdots, X_{n+1}]$ (NOT preserve degree)

Conversely, $f \in R[X_1, \cdots, X_n]$ of deg $= d$. $(f = f_0 + f_1 + \cdots + f_d)$

(preserve degree)

$f^* := X_{n+1}^d f_0 + X_{n+1}^{d-1} f_1 + \cdots + f_d = X_{n+1}^d f(X_1/X_{n+1}, \cdots, X_n/X_{n+1})$

$$\{\text{Forms in } R[X_1 \cdots X_{n+1}]\} \underset{\text{homogenizing}}{\overset{\text{dehomogenizing}}{\rightleftarrows}} \{\text{poly. in } R[X_1 \cdots X_n]\}$$ w.r.t. X_{n+1}

Example: $XY + Y^2 \xrightarrow{(\cdot)_*} x+1 \xrightarrow{(\cdot)^*} X+Y \xrightarrow{(\cdot)_*} x+1$

Basic facts :

Prop: (1). $(FG)_* = F_* G_*$; $(fg)^* = f^* g^*$

(2). $(f^*)_* = f$; $X_{n+1}^r (F_*)^* = F$ $(F \neq 0,\ X_{n+1}^r \| F)$

(3). $(F+G)_* = F_* + G_*$; $X_{n+1}^t (f+g)^* = X_{n+1}^r f^* + X_{n+1}^s g^*$

$(r = \deg g,\ s = \deg f,\ t = r+s - \deg(f+g))$.

Cor: 1) factoring form $F \underset{}{\overset{\text{up to power of } X_{n+1}}{\Longleftrightarrow}}$ factoring poly. F_*

2) F = form in $k[X,Y]$. $F \overset{k=\bar{k}}{=} \prod(\text{linear factors})$

Pf: prop (1) (2) $\Rightarrow$ 1).

Prop $k=\bar{k}$, $I \lhd k[x_1,\cdots,x_n]$, $V(I)=\{P_1,\cdots,P_N\}$ (finite). $\mathcal{O}_i := \mathcal{O}_{P_i}(\mathbb{A}^n)$. Then

$$\varphi : R := k[x_1,\cdots,x_n]/I \xrightarrow{\sim} \prod_{i=1}^{n} \underbrace{\mathcal{O}_i / I\mathcal{O}_i}_{=: R_i}$$

$$f \bmod I \longmapsto (f \bmod I\mathcal{O}_i).$$

Cor $\dim_k (k[x_1,\cdots,x_n]/I) = \sum_{i=1}^{N} \dim_k (\mathcal{O}_i / I\mathcal{O}_i)$

Cor If $V(I)=\{P\}$, then $k[x_1,\cdots,x_n]/I \xrightarrow{\sim} \mathcal{O}_P(\mathbb{A}^n)/I\mathcal{O}_P(\mathbb{A}^n)$.

Lem (Chinese Reminder Thm): $J_1,\cdots,J_N \lhd R$ ideals $J_i+J_j=R$ $\forall i\neq j$

Then $J := J_1\cdots J_N = J_1\cap\cdots\cap J_N$ and

$$R/J \cong \prod_{i=1}^{N} R/J_i$$

Pf: • inductively assume $J_2\cdots J_N = J_2\cap\cdots\cap J_N$

$R=(J_1+J_2)\cdots(J_1+J_N) \subseteq J_1 + J_2\cdots J_N \Rightarrow J=J_1\cap\cdots\cap J_N$

• WMA $N=2$.

$$R/J_1\cap J_2 \xrightarrow{\pi} R/J_1 \times R/J_2$$

$1\in J_1+J_2 \Rightarrow 1=e_2+e_1$ $\quad e_2\in J_1$ $\quad e_1\in J_2$ $\begin{pmatrix} e_1\equiv 1 \ (J_1) \\ e_2 \equiv 1 \ (J_2)\end{pmatrix}$

$\Rightarrow \forall (a+J_1, b+J_2) \in R/J_1 \times R/J_2$

$\Rightarrow \pi(ae_1+be_2) = (a+J_1, b+J_2) \Rightarrow \checkmark$

Pf of Prop:

$I_i := I(\{p_i\}) \Rightarrow \sqrt{I} = I_1 \cdots I_N \Rightarrow I_1^d \cdots I_N^d \subseteq I$ for some d.

$J_i := I_i / I \lhd R \quad (J_1^d \cdots J_N^d = 0 \text{ in } R)$

$I_i + I_j = k[x_1 \cdots x_n] \Rightarrow J_i + J_j = R$

$\Rightarrow R \cong R / J_1^d \cdots J_N^d \cong R/J_1^d \times \cdots \times R/J_N^d$

Rmk: $R/J_i^d = \dfrac{k[x]/I}{I_i^{d+1}/I} \cong k[x]/(I, I_i^d)$

WANTS: $k[x]/(I, I_i^d) \cong \mathcal{O}_i / I\mathcal{O}_i$

$\bar{f} \mapsto \bar{f}$

$\overline{\text{Inj}}$:

- $\forall f \in I\mathcal{O}_i \cap k[x] \Rightarrow gf \in I$ with $g(p) \neq 0$

$\Downarrow$

$(g) + I_i^d = k[x]$

$\Downarrow$

$\exists h$ s.t. $hg - 1 \in I_i^d$

$\Rightarrow f \equiv hgf \mod I_i^d$

$\Rightarrow f \in (I, I_i^d)$

Surj: $\forall \frac{f}{g} \in \mathcal{O}_i \quad (I\mathcal{O}_i \supseteq I_i^d \mathcal{O}_i)$

$hg - 1 \in I_i^d \subseteq I_i^d \mathcal{O}_i \subseteq I\mathcal{O}_i$ $\qquad hg - 1 \in I_i^d$

$\Rightarrow \frac{f}{g} \equiv fh \mod I\mathcal{O}_i \Rightarrow$ surj.